

# Fermion Mass Generation in the $D$ -dimensional Thirring Model as a Gauge Theory

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## Abstract

Based on the Schwinger-Dyson (SD) equation, the fermion mass generation is further studied in the  $D(2 < D < 4)$ -dimensional Thirring model as a gauge theory previously proposed. By using a certain approximation to the kernel, we analytically obtained explicit form of the dynamical mass of fermion and the critical line in  $(N, 1/g)$  space, where  $N$  is the number of fermions and  $g$  is the dimensionless vector-type four-fermion coupling constant. This analytical result is confirmed by the numerical solution for the SD equation with exact form of the kernel in  $(2+1)$  dimensions.

## §1. Introduction

Through the studies of the dynamical origin of the quarks/leptons mass, it was discovered<sup>1), 2), 3)</sup> that the dynamics inducing the *large anomalous dimension* of the operator  $\bar{\psi}\psi$  is strongly required.<sup>1)</sup> In particular, the scalar/pseudoscalar-type four-fermion interactions with the gauge interaction (gauged Nambu-Jona-Lasinio model<sup>5), 6), 2)</sup>) have played a very important role as a renormalizable model in (3+1) dimensions ( $D = 4$ ).<sup>7)</sup> It has also been shown<sup>7)</sup> that the phase structure of such a gauged NJL model in (3+1) dimensions is quite similar to that of the  $D(2 < D < 4)$ -dimensional four-fermion theory of scalar/pseudoscalar-type (without gauge interactions),<sup>8)</sup> often called Gross-Neveu model,<sup>9)</sup> which is renormalizable in  $1/N$  expansion.<sup>10)</sup>

Recently a model with only the vector-type four-fermion interaction has been studied as another candidate for the scenario of the fermion dynamical mass generation,<sup>11), 12), 13), 14), 15)</sup> namely, the  $D(2 < D < 4)$ -dimensional Thirring model<sup>16)</sup> with  $N$  fermions. If we naively treated this model with the usual gap equation made by  $1/N$ -leading diagram, we would conclude no fermion mass generation. Nevertheless, we can obtain the dynamical mass of fermion by solving the SD equation with the rainbow diagram of the composite vector boson which is induced in  $1/N$ -leading order.<sup>11)</sup> Then the dynamical mass has a nontrivial dependence on  $1/N$  as a consequence of the SD equation similar to the case of QED in (2+1) dimensions.<sup>17)</sup>

In our previous paper<sup>12)</sup> we stressed that the composite vector boson is actually a dynamical gauge boson corresponding to the hidden local symmetry (HLS)<sup>18)</sup> being broken spontaneously, which means that the Lagrangian can be rewritten into the manifest gauge symmetric form. Combined with the nonlocal gauge fixing,<sup>19)</sup> this manifest gauge degree of freedom plays an important role in construction of the SD equation to keep the consistency between the ladder approximation and the Ward-Takahashi identity. Based on this SD equation, we performed the proof that the fermion dynamical mass is actually generated at finite value of  $N$  ( $N$ : number of four-component fermions) in  $D(2 < D < 4)$  dimensions,<sup>\*)</sup> through the context of Ref. 20). Moreover we obtained explicit form of the mass function and a critical value of  $N$  in the limit of  $g \rightarrow \infty$  in (2+1) dimensions, where  $g$  is the dimensionless four-fermion coupling constant of the Thirring model.

Further studies were done by several authors.<sup>13), 14), 15)</sup> The author of Ref. 13) derived the explicit form of the critical value of  $N$  for finite  $g$  (denoted as  $N_{\text{cr}}(g)$ ) in the  $D$ -dimensional Thirring model by using the inversion method instead of the SD equation. The explicit form

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<sup>\*)</sup> The original proof in Ref. 12) includes the case of  $D = 2$ , although the exact solution<sup>16)</sup> seems to tell us no mass generation. In Appendix A we will show why this difference occurs.

of  $N_{\text{cr}}(g)$  is somewhat different from the one given in Ref. 13) and this paper (See §4 \*)), although they agree with each other qualitatively as to the existence of the critical  $N$ . In Refs. 14), 15) the lattice simulation were performed in (2+1) dimensions, and the results seem to support those of analytical studies.<sup>12), 13)</sup>

In this paper we first recapitulate the formalism of Ref. 12) for the Thirring model as a gauge theory and the SD equation in the nonlocal gauge. We then give a revised proof of the existence of a nontrivial solution, since the previous proof<sup>12)</sup> was based on an implicit assumption and thus was not complete.

Next we show explicit form of the analytical solution in the limit of  $g \rightarrow \infty$  in  $D(2 < D < 4)$  dimensions not restricted to (2+1) dimensions. By introducing a certain ansatz for the kernel of the SD equation, we further obtain explicit form of the analytical solution for finite value of  $g$  not restricted to  $g \rightarrow \infty$ . The validity of the ansatz will be checked by the numerical solution in the case of (2+1) dimensions. The analytical solutions for the finite coupling constant will provide us with the information for investigating the phase structure, i.e., the critical line in  $(N, 1/g)$  plane, the beta function, the anomalous dimension, etc..

This paper is organized as follows. In §2 we briefly summarize the HLS and nonlocal gauge fixing procedure. In the first subsection of the §3 the review will be continued to introduce how to construct the SD equation, and in the last subsection we will improve the existence proof of a nontrivial solution of the SD equation as a supplement to the previous one.<sup>12)</sup> In §4 the analytical calculations for the SD equation in  $D(2 < D < 4)$  dimensions will be done to obtain explicit form of the solutions used to investigate the properties of the model. §5 will be devoted to the comparison between the analytical and the numerical solutions in (2+1) dimensions. The conclusion and discussion will be given in §6.

## §2. Hidden Local Symmetry

### 2.1. Hidden Local Symmetry in the Thirring Model

In this and the next sections we would like to review briefly the formulation used in Ref. 12) to investigate the dynamical mass generation of fermion in the  $D$ -dimensional Thirring model.

Let us start with the following Lagrangian (the Thirring Model):

$$\mathcal{L}_{\text{Thi}} = \sum_{a=1}^N \bar{\Psi}_a i \not{\partial} \Psi_a - \frac{G}{2N} \left( \sum_{a=1}^N \bar{\Psi}_a \gamma^\mu \Psi_a \right)^2, \quad (2.1)$$

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\*) Although we give only the critical value of  $g$ , as  $g_{\text{cr}}(N)$ , we can read  $N_{\text{cr}}(g)$  by inverting  $g = g_{\text{cr}}(N_{\text{cr}}(g))$  with respect to  $N_{\text{cr}}(g)$ .

where it is supposed that each of Dirac gamma matrices is represented formally as  $4 \times 4$  matrix even in  $D(2 < D < 4)$  dimensions, and the subscript  $a$  runs over the *color* of fermions. (Hereafter we will suppress summation symbol,  $\sum_{a=1}^N$ .) It is well known that this Lagrangian can be rewritten with the vector auxiliary field,  $\tilde{A}_\mu$ , such as

$$\mathcal{L}_{\text{Thi}} = \bar{\Psi}_a i\gamma^\mu \left( \partial_\mu - \frac{i}{\sqrt{N}} \tilde{A}_\mu \right) \Psi_a + \frac{1}{2G} \tilde{A}_\mu^2, \quad (2.2)$$

where it is easy to see that  $\tilde{A}_\mu(x)$  is related to fermions,

$$\tilde{A}_\mu = -\frac{G}{\sqrt{N}} \bar{\Psi}_a \gamma_\mu \Psi_a, \quad (2.3)$$

from the equation of motion. Eq.(2.2) has no local symmetry associated with  $\tilde{A}_\mu$  due to the “mass term” appearing in the last of Eq.(2.2). Nevertheless, we can rewrite Eq.(2.1) or equivalently Eq.(2.2) so that it actually has *the hidden local symmetry* (HLS)<sup>(21), (22), (23), (18), (12)</sup> associated with the gauge field  $A_\mu$  which is introduced through the following field redefinition:

$$\Psi_a(x) = u^\dagger(x) \psi_a(x) = e^{-i\phi(x)} \psi_a(x), \quad (2.4a)$$

$$\tilde{A}_\mu(x) = i\sqrt{N} u^\dagger \cdot D_\mu u = A_\mu - \sqrt{N} \partial_\mu \phi, \quad (2.4b)$$

where the field  $u(x) \equiv e^{i\phi(x)}$  can be recognized as the nonlinearly-realized basis of the NG boson field  $\phi(x)$  accompanying the spontaneous symmetry breaking of the hidden local symmetry,  $U(1)_{\text{hidden}}$ . Thus, using the fields defined above, the Lagrangian (2.1) is rewritten into the following form,

$$\begin{aligned} \mathcal{L}_{\text{Thi}} &= \bar{\psi}_a i \not{D} \psi_a - \frac{N}{2G} (u^\dagger \cdot D_\mu u)^2, \\ &= \bar{\psi}_a i\gamma^\mu \left( \partial_\mu - \frac{i}{\sqrt{N}} A_\mu \right) \psi_a + \frac{1}{2G} (A_\mu - \sqrt{N} \partial_\mu \phi)^2, \end{aligned} \quad (2.5)$$

in which the HLS is realized manifestly at the Lagrangian level:

$$\psi_a \mapsto \psi'_a = e^{i\alpha} \psi_a, \quad A_\mu \mapsto A'_\mu = A_\mu + \sqrt{N} \partial_\mu \alpha, \quad \phi \mapsto \phi' = \phi + \alpha. \quad (2.6)$$

“Gauge equivalence” between Eq.(2.2) and (2.5) can be shown by taking particular choice of the gauge transformation in Eq.(2.6) as  $\alpha(x) = -\phi(x)$ , called the unitary gauge. If we further identify  $\psi'_a = \Psi_a$  and  $A'_\mu = \tilde{A}_\mu$  under such a gauge, Eq.(2.6) is identical to Eqs.(2.4).

Why should the existence of the HLS be emphasized? The reasons are the following: First, coupled with the BRS formalism,<sup>(24)</sup> this local symmetry might enable us to prove the S-matrix unitarity more straightforwardly. Secondly, actual calculations, particularly

loop calculations, are generally hopeless in the unitary gauge, while the HLS provides us to take the most appropriate gauge for our purpose. For these reasons, the BRS gauge fixing procedure should be applied to Eq.(2.5) instead of the unitary gauge. Moreover, we should introduce a derivative- (momentum-) dependent gauge fixing parameter  $\xi(\partial^2)$  ( $\xi(k^2)$ ), i.e., the nonlocal gauge fixing<sup>19)</sup> since it is needed to guarantee the Ward-Takahashi identity in the ladder SD equation (see §3). It has also been proven that we can construct the BRS symmetric Lagrangian even for the nonlocal gauge.<sup>12)</sup>

Here we adopt the  $R_\xi$  gauge fixing term such as

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2} \left( \partial_\mu A^\mu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi \right) \frac{1}{\xi(\partial^2)} \left( \partial_\nu A^\nu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi \right), \quad (2.7)$$

where  $\xi(\partial^2)$  is a nonlocal (momentum dependent) parameter. Therefore the total Lagrangian is given by

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{Thi}} + \mathcal{L}_{\text{GF}} = \mathcal{L}_{\psi,A} + \mathcal{L}_\phi, \quad (2.8)$$

$$\mathcal{L}_{\psi,A} = \bar{\psi}_a i \not{D} \psi_a + \frac{1}{2G} (A_\mu)^2 - \frac{1}{2} \partial_\mu A^\mu \left( \frac{1}{\xi(\partial^2)} \partial_\nu A^\nu \right), \quad (2.9)$$

$$\mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2G} (\xi(\partial^2) \phi) \phi, \quad (2.10)$$

where the  $\phi$  was rescaled as  $\sqrt{N/G} \phi \mapsto \phi$ . In this gauge fixing the fictitious NG boson  $\phi$  is completely decoupled independently whether  $\xi$  is nonlocal or not. Equation (2.9) appears as if we added the “covariant gauge fixing term” to Eq.(2.2). However, without introducing the HLS, there is no reason why we should add such a term. Such a confusion was actually made by some authors<sup>25), 11)</sup> who happened to arrive at the Lagrangian having the same form as Eq.(2.2) in the case of constant  $\xi$ . Here we stress again that this Lagrangian (2.9), whether the gauge parameter is nonlocal or not, can only be justified through the HLS in the  $R_\xi$  gauge.

### §3. Schwinger-Dyson Equation and Dynamical Mass Generation

The purposes of this section are to briefly review how to construct the SD equation of the  $D$ -dimensional Thirring Model<sup>12)</sup> and to improve the existence proof of a nontrivial solution of the SD equation given in Ref. 12).

### 3.1. Schwinger-Dyson Equation in the nonlocal $R_\xi$ gauge

The SD equation for the fermion full propagator  $S(p) = i[A(-p^2)\not{p} - B(-p^2)]^{-1}$ , with  $B(p)$  being the order parameter of the chiral symmetry breaking, is written as follows:

$$\begin{aligned} & (A(-p^2) - 1)\not{p} - B(-p^2) \\ &= -\frac{1}{N} \int \frac{d^D q}{i(2\pi)^D} \gamma_\mu \frac{A(-q^2)\not{q} + B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} \Gamma_\nu(p, q) iD^{\mu\nu}(p - q), \end{aligned} \quad (3.1)$$

where  $\Gamma_\nu(p, q)$  and  $D_{\mu\nu}(p - q)$  denote the full vertex function and the full propagator of the dynamical gauge boson, respectively. As indicated in our previous paper,<sup>12)</sup> the R.H.S. of Eq.(3.1) is  $O(1/N)$ , which implies that there is no nontrivial solution in the  $1/N$  expansion. Thus we should solve it and find the solution which depends on  $1/N$  in a non-analytic way similar to the case of the QED<sub>3</sub>.<sup>17)</sup>

We should apply some appropriate approximations to Eq.(3.1) so as to reduce it to the soluble integral equation for the mass function  $M(-p^2) = B(-p^2)/A(-p^2)$ . Approximations adopted here are the following: First, bare vertex (ladder) approximation is used to  $\Gamma^\mu(p, q)$ , i.e.,  $\Gamma^\mu(p, q) = \gamma^\mu$ . Secondly, only the one-loop diagram of the bare fermion propagators  $i/\not{p}$  is taken to evaluate the vacuum polarization tensor  $\Pi_{\mu\nu}(-k^2)$ . This approximation may be consistent with both  $A(-p^2) = 1$  in the nonlocal gauge fixing and the bifurcation theory used in later sections, which reduces the SD equation as a nonlinear integral equation to the linear one with respect to  $B(-p^2)$ .

The vacuum polarization tensor in these approximations is given by

$$\Pi_{\mu\nu}(k) \equiv \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(-k^2), \quad (3.2)$$

$$\Pi(-k^2) = -C_D^{-1} \cdot (-k^2)^{\frac{D-2}{2}}, \quad C_D^{-1} \equiv \frac{2\text{tr}I}{(4\pi)^{D/2}} \Gamma(2 - \frac{D}{2}) B(\frac{D}{2}, \frac{D}{2}), \quad (3.3)$$

where  $\text{tr}I$ , equal to four by an assumption argued in §2, is the trace of unit matrix in spinor indices, and thus the gauge boson full propagator is as follows:

$$iD_{\mu\nu}(k) = id(-k^2) \left[ g_{\mu\nu} - \eta(-k^2) \frac{k_\mu k_\nu}{k^2} \right], \quad (3.4)$$

$$d(-k^2) = \frac{1}{G^{-1} - \Pi(-k^2)}, \quad \eta(-k^2) = \frac{\xi(-k^2)\Pi(-k^2) - k^2}{\xi(-k^2)G^{-1} - k^2}. \quad (3.5)$$

Then the SD equation (3.1) is reduced to the following coupled equations for  $A(-p^2)$  and  $B(-p^2)$ :

$$A(-p^2) - 1 = \frac{1}{Np^2} \int \frac{d^D q}{i(2\pi)^D} \frac{A(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)}$$

$$\times d(-k^2) \left[ \{ \eta(-k^2) + 2 - D \} (p \cdot q) - \frac{2(k \cdot p)(k \cdot q)}{k^2} \eta(-k^2) \right], \quad (3.6a)$$

$$B(-p^2) = -\frac{1}{N} \int \frac{d^D q}{i(2\pi)^D} \frac{B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} \times d(-k^2)[D - \eta(-k^2)], \quad (3.6b)$$

where  $k_\mu = p_\mu - q_\mu$ .

It is generally difficult to deal with the coupled equations. We follow the nonlocal gauge proposed by Georgi et al.<sup>19), 26)</sup> which reduces the coupled SD equations into a single equation for  $B(-p^2)$ . Due to the freedom of gauge choice of  $\xi(-k^2)$ , the R.H.S. of Eq.(3.6a) can be set to zero, which implies  $A(-p^2) \equiv 1$ . In such a nonlocal gauge  $B(-p^2)$  itself is a mass function, i.e.,  $M(-p^2) = B(-p^2)$ .

Requiring  $A(-p^2) \equiv 1$  in the nonlocal gauge, Eqs.(3.6) are translated into the following forms (hereafter the Euclidean notation is used):

$$0 = \int_0^\pi d\theta \sin^D \theta \left[ \frac{1}{D-1} \frac{d}{dk^2} \left\{ d(k^2)(\eta(k^2) + D - 2) \right\} + \frac{\eta(k^2)d(k^2)}{k^2} \right], \quad (3.7a)$$

$$B(p^2) = \frac{1}{N} \int_0^{\Lambda^{D-2}} d(q^{D-2}) K(p, q; G) \frac{q^2 B(q^2)}{q^2 + B^2(q^2)}, \quad (3.7b)$$

where the kernel  $K(p, q; G)$  is given by

$$K(p, q; G) = \frac{1}{2^{D-1} \pi^{\frac{(D+1)}{2}} (D-2) \Gamma(\frac{D-1}{2})} \int_0^\pi d\theta \sin^{D-2} \theta \times d(k^2)[D - \eta(k^2)]. \quad (3.8a)$$

$$(k^2 = p^2 + q^2 - 2pq \cos \theta.) \quad (3.8b)$$

The kernel Eq.(3.8a) has the following properties; symmetry under the exchange among first two arguments as  $K(p, q; G) = K(q, p; G)$ , positivity and finiteness summarized as  $0 < K(p, q; G) < \infty$  for  $G > 0$ . Here the ultraviolet momentum cut-off  $\Lambda$  is needed since the integral of the R.H.S. of Eq.(3.7b) diverges according to the power counting. Now the role of the equations (3.7) appears to be separated. Eq.(3.7a) determines the momentum dependence of  $\eta(k^2)$  ( $\xi(k^2)$ ) as

$$\eta(k^2) = -\frac{(D-2)}{k^{2(D-1)}d(k^2)} \int_0^{k^2} d\zeta \zeta^{D-1} d'(\zeta), \quad (3.9)$$

which is obtained by integrating the R.H.S. with respect to  $k^2$ . Eq.(3.7b) is therefore reduced to the integral equation with respect to the only one function,  $B(p^2)$ .

### 3.2. Existence Proof of a Nontrivial Solution and the Critical Line

From now on we reconsider the existence proof of a nontrivial solution of the SD equation (3.7b), based on the bifurcation theory discussed in Refs. 20), 27) and briefly summarized in Appendix A.

Eq.(3.7b) always has a trivial solution  $B(p^2) \equiv 0$ . We are often interested in the vicinity of the phase transition point where the nontrivial solution also starts to exist without gap. Such a bifurcation point is identified by the existence of an infinitesimal solution  $\delta B(p^2)$  around the trivial one  $B(p^2) \equiv 0$ .<sup>27)</sup> Then we obtain the linearized equation for  $\delta B(p^2)$ :

$$\delta B(p^2) = \frac{1}{N} \int_{m^{D-2}}^{\Lambda^{D-2}} d(q^{D-2}) K(p, q; G) \delta B(q^2), \quad (3.10)$$

where the IR cut-off  $m$  is introduced. It is enough for us to show the existence of a nontrivial solution of the bifurcation equation (3.10).<sup>20)</sup> Particularly, we can obtain the exact phase transition point where the bifurcation takes place. Since the solution is normalized as  $m = \delta B(m^2)$ ,  $m$  is nothing but the dynamically generated fermion mass.

Rescaling  $p = \Lambda x^{\frac{1}{D-2}}$  and  $\delta B(p^2) = \Lambda \Sigma(x)$ , Eq.(3.10) can be rewritten as follows:

$$\Sigma(x) = \frac{1}{N} \int_{\sigma_m}^1 dy \widetilde{K}(x, y; g) \Sigma(y), \quad (3.11)$$

where  $\sigma_m \equiv (m/\Lambda)^{D-2}$  ( $\sigma_m < 1$ ), the dimensionless four-fermion coupling constant  $g \equiv G\Lambda^{D-2}$  and the dimensionless kernel is defined by

$$\widetilde{K}(x, y; g) \equiv K(x^{\frac{1}{D-2}}, y^{\frac{1}{D-2}}; g). \quad (3.12)$$

As we mentioned in the end of the previous subsection, the kernel  $\widetilde{K}(x, y; g)$  is positive, symmetric and finite:

$$0 < \widetilde{K}(x, y; g) = \widetilde{K}(y, x; g) < \infty, \quad \text{for } x, y \in [\sigma_m, 1], g \geq 0. \quad (3.13)$$

These are the most important properties for the existence proof of a nontrivial solution.<sup>20)</sup>

Here let us consider the linear integral equation as an eigenvalue problem:

$$\phi(x) = \frac{1}{\lambda} \int_{\sigma_m}^1 dy \widetilde{K}(x, y; g) \phi(y), \quad (3.14)$$

whose eigenvalues and eigenfunctions are denoted by  $\lambda_n(g, \sigma_m)$  ( $|\lambda_n| \geq |\lambda_{n+1}|$ ;  $n = 1, 2, \dots$ ) and  $\phi_n(x)$ , respectively. The kernel  $\widetilde{K}(x, y; g)$  is a symmetric one and hence satisfies the following property:

$$\sum_{n=1}^{\infty} \lambda_n^2(g, \sigma_m) = \int_{\sigma_m}^1 \int_{\sigma_m}^1 dx dy [\widetilde{K}(x, y; g)]^2 < \infty. \quad (3.15)$$



The R.H.S. of Eq.(3.15) gives the upper bound for each eigenvalues  $\lambda_n(g, \sigma_m)$ , so these are finite. Furthermore, using the positivity of the symmetric kernel (see Eq.(3.13)), it is easily proven that the maximal eigenvalue  $\lambda_1(g, \sigma_m)$  is always positive and the corresponding eigenfunction  $\phi_1(x)$  has a definite sign (nodeless solution). Moreover, we can also show that  $\lambda_1(g, \sigma_m)$  is monotonically decreasing function for  $\sigma_m$ , i.e.,  $\lambda_1(g, \sigma_m) < \lambda_1(g, \sigma'_m)$  for  $\sigma_m > \sigma'_m$ , as proven in Appendix B. This behavior of the eigenvalue for  $\sigma_m$  was implicitly assumed in the previous proof.<sup>12)</sup>

In the bifurcation equation (3.11) this implies the following: *If  $N$  is smaller than the maximal eigenvalue of the kernel at  $\sigma_m = 0$  :  $N \leq \lambda_1(g, 0)$ , then there exists a value of  $\sigma_m$  at which  $N = \lambda_1(g, \sigma_m)$  and the corresponding nontrivial nodeless solution  $\Sigma(x) = \phi_1(x)$  besides a trivial one.* The above statement means that the parameter  $\sigma_m$  corresponds to the dynamically generated mass  $m = \Lambda(\sigma_m)^{\frac{1}{D-2}}$ . Moreover,  $N_{\text{cr}}(g)$  introduced as

$$N_{\text{cr}}(g) = \lambda_1(g, \sigma_m = 0) \quad (3.16)$$

determines the critical line in  $(N, 1/g)$  plane and separating it by two parts which are corresponding to the broken and the symmetric phase, since there is no non-zero solution for  $N > N_{\text{cr}}(g)$  by definition. Existence of the critical line,  $N = N_{\text{cr}}(g)$  or  $g = g_{\text{cr}}(N)$ , in the two-parameter space is somewhat analogous to that in the gauged NJL model.<sup>6)</sup>

Finally, we can point out something more about the critical line by using the explicit form of the kernel. From Eq.(3.8) it can be seen that the kernel vanishes when  $g = 0$ . This means that  $\lambda_1(g = 0, \sigma_m) = 0$ , therefore the critical line starts from  $(0, \infty)$ . On the other hand, the kernel remains finite when taking  $g \rightarrow \infty$  owing to the IR cut-off, which concludes that the critical line ends at  $(N_{\text{cr}}(\infty), 0)$  with non-zero  $N_{\text{cr}}(\infty)$ .\*)

## §4. Analytical Solutions for $D(2 < D < 4)$ Dimensions

Without solving the SD equation (3.7) explicitly, it has been proven that there exists the critical line in  $(N, 1/g)$  plane, which starts from  $(0, \infty)$  to  $(N_{\text{cr}}(\infty), 0)$ . However, we have to find an explicit form of the solution to know more about the properties of the model, which are, for example, momentum dependence of the mass function, the beta function and the anomalous dimension of the operator  $\bar{\psi}\psi$ , etc.. Therefore we attempt to solve the SD equation only for the particular regions of the coupling constant, since the integral for the kernel, Eq.(3.8a), cannot be performed for general value of  $g$ . Furthermore we restrict

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\*) At this stage we cannot yet conclude that  $N_{\text{cr}}(\infty)$  is finite since  $\int_0^1 \int_0^1 dx dy \tilde{K}(x, y; \infty)^2 = \infty$ . However, the explicit calculation in the next section tells us that  $N_{\text{cr}}(\infty)$  is actually finite.

ourselves to analyze the bifurcation equation (3.10) instead of Eq.(3.7b), since it is difficult to solve Eq.(3.7b) being a nonlinear integral equation. Thus we will investigate the properties of the model near the critical line in  $(N, 1/g)$  plane, in which we are most interested.

#### 4.1. Analytical Solution : $g \rightarrow \infty$ case

First we consider that  $g$  is infinity, which is identical to the vanishing limit of the dynamical gauge boson mass proportional to  $1/g$ . In this case the spontaneously broken  $U(1)_{\text{hidden}}$  becomes a manifest symmetry of physical states, therefore we expect that the result is similar to that of  $D$ -dimensional QED, which we will discuss in the end of this subsection.

The functions of the propagator of  $A_\mu$  defined in Eq.(3.4) have now the following form.

$$d(k^2) = \frac{1}{-\Pi(k^2)} = C_D k^{2-D}, \quad (4.1)$$

$$\eta(k^2) = \frac{(D-2)^2}{D}, \quad (4.2)$$

where  $\eta(k^2)$  is determined by Eq.(3.9) and is independent of the momentum  $k^2$ , which means that the gauge fixing parameter  $\xi(k^2)$  is actually *nonlocal* according to Eq.(3.5):

$$\xi(k^2) = -\frac{(D-1)(D-4)}{D} C_D k^{4-D}. \quad (4.3)$$

Furthermore, the integral kernel of the SD equation Eq.(3.8) can be calculated exactly as

$$\begin{aligned} K(p, q; \infty) &= K_D \left( \frac{2}{p+q+|p-q|} \right)^{D-2} \\ &= K_D \left[ \theta(p^{D-2} - q^{D-2}) \frac{1}{p^{D-2}} + \theta(q^{D-2} - p^{D-2}) \frac{1}{q^{D-2}} \right], \end{aligned} \quad (4.4)$$

$$K_D \equiv \frac{(D-1)\Gamma(D)}{D(D-2)\Gamma(2-\frac{D}{2})(\Gamma(\frac{D}{2}))^3}. \quad (4.5)$$

By using the same rescaling as Eq.(3.11), such as  $x = (p/\Lambda)^{D-2}$ ,  $y = (q/\Lambda)^{D-2}$ ,  $\sigma_m = (m/\Lambda)^{D-2}$  and  $\Sigma(x) = B(p^2)/\Lambda$ , we can rewrite the SD equation as

$$\Sigma(x) = \frac{K_D}{N} \int_{\sigma_m}^1 dy \left[ \theta(x-y) \frac{1}{x} + \theta(y-x) \frac{1}{y} \right] \Sigma(y). \quad (4.6)$$

As usually done for the SD equation of QED, the solution of this integral equation is obtained by translating Eq.(4.6) into the following differential equation with the boundary conditions and the normalization condition:

$$\frac{d}{dx} \left( x^2 \frac{d\Sigma(x)}{dx} \right) + \frac{N_{\text{cr}}^\infty}{4N} \Sigma(x) = 0, \quad (4.7)$$

$$\Sigma'(\sigma_m) = 0, \quad (\text{IR B.C.}) \quad (4.8a)$$

$$(x\Sigma)'|_{x=1} = 0, \quad (\text{UV B.C.}) \quad (4.8b)$$

$$\Sigma(\sigma_m) = \sigma_m^{1/(D-2)}, \quad (\text{Normalization Condition}) \quad (4.8c)$$

where,

$$N_{\text{cr}}^\infty \equiv 4K_D = \frac{4(D-1)\Gamma(D)}{D(D-2)\Gamma(2-\frac{D}{2})(\Gamma(\frac{D}{2}))^3} \quad (4.9)$$

is the critical value of  $N$  which corresponds to  $N_{\text{cr}}(\infty)$  appearing in the subsection 3.2., as will be shown later.

The solution of Eq.(4.7) with the IR B.C. as well as the normalization condition has different form for each value of  $N$ :

- Region I ( $N < N_{\text{cr}}^\infty$ ):

$$\begin{aligned} \Sigma(x) &= \frac{\sigma_m^{\frac{1}{D-2}}}{\sin\left(\frac{\omega}{2}\delta\right)} \left(\frac{\sigma_m}{x}\right)^{\frac{1}{2}} \sin\left\{\frac{\omega}{2}\left[\ln\frac{x}{\sigma_m} + \delta\right]\right\}, \\ \omega &= \sqrt{\frac{N_{\text{cr}}^\infty}{N} - 1}, \quad \delta = 2\omega^{-1}\tan^{-1}\omega. \end{aligned} \quad (4.10)$$

- Region II ( $N = N_{\text{cr}}^\infty$ ):

$$\Sigma(x) = \sigma_m^{\frac{1}{D-2}} \left(\frac{\sigma_m}{x}\right)^{\frac{1}{2}} \left[\frac{1}{2}\ln\frac{x}{\sigma_m} + 1\right]. \quad (4.11)$$

- Region III ( $N > N_{\text{cr}}^\infty$ ):

$$\begin{aligned} \Sigma(x) &= \frac{\sigma_m^{\frac{1}{D-2}}}{\sinh\left(\frac{\omega'}{2}\delta'\right)} \left(\frac{\sigma_m}{x}\right)^{\frac{1}{2}} \sinh\left\{\frac{\omega'}{2}\left[\ln\frac{x}{\sigma_m} + \delta'\right]\right\}, \\ \omega' &= \sqrt{1 - \frac{N_{\text{cr}}^\infty}{N}}, \quad \delta' = 2\omega'^{-1}\tanh^{-1}\omega'. \end{aligned} \quad (4.12)$$

Since only the solution of Region I can satisfy the UV B.C. for  $\sigma_m \neq 0$ , we conclude that the dynamical mass is generated only in case of  $N < N_{\text{cr}}^\infty$ , which means  $N_{\text{cr}}^\infty$  is actually the critical value separating the symmetric phase and the broken one. As already mentioned in our previous paper,<sup>12)</sup> in (2+1) dimensions the critical value  $N_{\text{cr}}^\infty = 128/3\pi^2$  is identical to the one in QED<sub>3</sub>.<sup>26)</sup> Furthermore, the UV B.C. Eq.(4.8b) gives the relation between  $N$  and  $\sigma_m = (m/\Lambda)^{(D-2)}$  as

$$\frac{\omega}{2}\left[\ln\frac{1}{\sigma_m} + 2\delta\right] = n\pi, \quad n = 1, 2, \dots, \quad (4.13)$$

where the solution with  $n = 1$  corresponds to the nodeless (ground state) solution whose scaling behavior is read from Eq.(4.13):

$$\frac{m}{\Lambda} = e^{\frac{2\delta}{D-2}} \exp \left[ -\frac{2\pi}{(D-2)\sqrt{N_{\text{cr}}^\infty/N-1}} \right]. \quad (4.14)$$

As was done in Refs. 28), 12), we recognize it as determining the cut-off dependence of  $N = N(\Lambda)$ , by which we can have a finite  $m$  when  $\Lambda$  is taken to be infinite (continuum limit). In fact, if  $N(\Lambda)$  goes to  $N_{\text{cr}}^\infty$  continuously when  $\Lambda \rightarrow \infty$ , it is possible to keep  $m$  finite, and hence the beta function which determines the behavior of  $N(\Lambda)$  with respect to  $\Lambda$  is derived as

$$\beta_\Lambda(N) \equiv \Lambda \frac{\partial N(\Lambda)}{\partial \Lambda} = \frac{(D-2)}{\pi N_{\text{cr}}^\infty} N^2 \left( \frac{N_{\text{cr}}^\infty}{N} - 1 \right)^{\frac{3}{2}}, \quad (4.15)$$

where this form is valid for  $N$  being nearly below  $N_{\text{cr}}^\infty$ . Eq.(4.15) implies that the critical value  $N_{\text{cr}}^\infty$  is a UV fixed point on which the continuum theory will be determined. Furthermore the anomalous dimension of the operator  $\bar{\psi}\psi$ , which is relatively large ( $\gamma_{\bar{\psi}\psi} \sim O(1)$ ) in models inducing the dynamical mass generation, is obtained by

$$\begin{aligned} \gamma_{\bar{\psi}\psi}(N) &= \frac{d \ln \langle \bar{\psi}\psi \rangle_\Lambda}{d \ln \Lambda} \\ &= \frac{D-2}{2} + 2N\beta_\Lambda(N) \\ &= \frac{D-2}{2} \left[ 1 - \frac{4N^3}{\pi N_{\text{cr}}^\infty} \left( \frac{N_{\text{cr}}^\infty}{N} - 1 \right)^{\frac{3}{2}} \right]. \end{aligned} \quad (4.16)$$

In Eq.(4.16) the vacuum expectation value  $\langle \bar{\psi}\psi \rangle_\Lambda$  is given as

$$\langle \bar{\psi}\psi \rangle_\Lambda = -\frac{32}{(4\pi)^{\frac{D}{2}} (D-2) \Gamma(\frac{D}{2}) N_{\text{cr}}^\infty} \cdot N^2 \cdot m^{\frac{D}{2}} \Lambda^{\frac{D-2}{2}}, \quad (4.17)$$

where we used the following expression for  $\langle \bar{\psi}\psi \rangle_\Lambda$  in the bifurcation theory:

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_\Lambda &= -\frac{2\text{tr}I}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} \frac{N}{\int_0^\Lambda dp p^{D-1} \frac{B(p)}{p^2 + B^2(p)}} \\ &\simeq -\frac{2\text{tr}I}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} \frac{N}{\int_m^\Lambda dp p^{D-3} B(p)}, \end{aligned} \quad (4.18)$$

and substituted  $B(p)$  into the solution in Region I. As is expected, Eq.(4.16) tells us that the anomalous dimension  $\gamma_{\bar{\psi}\psi}(N)$  is actually large, especially  $\gamma_{\bar{\psi}\psi}(N) \simeq (D-2)/2$  in the vicinity of the critical point.

The above renormalization group functions are obtained along with the horizontal line in  $(N, 1/g)$  plane, which is specified by  $1/g \equiv 0$ . However, it might not be natural to give

the cut-off dependence to  $N$  which should be an integer number. Thus we hope to know the solution for a finite  $g$  to attach the  $\Lambda$ -dependence not to  $N$  but to  $g$ , although we cannot solve the SD equation exactly for any  $g$ . In the rest of this section we will solve the SD equation Eq.(3.10) for finite  $g$  though restricted region.

#### 4.2. Analytical Solution : $g \gg 1$ case

First we attempt to find the next-to-leading order solution in  $1/g$  expansion, which means that we expand Eq.(3.10) in  $1/g$  up to  $O(1/g)$ . In this expansion the integrand of Eq.(3.8) is reduced to

$$\eta(k^2) = \frac{(D-2)^2}{D} \left\{ 1 - (D-1)C_D G^{-1} k^{2-D} \right\} + O(G^{-2}), \quad (4.19)$$

$$d(k^2)[D - \eta(k^2)] = \frac{4(D-1)C_D}{D} \cdot \frac{1}{k^{D-2} + f_D/G} + O(G^{-2}), \quad (4.20)$$

where

$$f_D \equiv \frac{(4-D)D}{4} C_D \quad (4.21)$$

is a value of  $O(1)$  in  $2 < D < 4$ . Eq.(4.20) seems to have a simple form, but we use the ansatz

$$k^2 = p^2 + q^2 - 2pq \cos \theta \simeq \max(p^2, q^2) \quad (4.22)$$

for Eq.(3.8b) to calculate the integral kernel analytically. This ansatz will be checked by the numerical study in §5 even in (2+1) dimensions. By substituting Eq.(4.22) into Eq.(4.20), we can evaluate Eq.(3.8a) as

$$K(p, q; G) = K_D \left[ \theta(p^{D-2} - q^{D-2}) \frac{1}{p^{D-2} + f_D/G} + (p \leftrightarrow q) \right], \quad (4.23)$$

where  $K_D$  is defined by Eq.(4.5). Similarly to the previous subsection, by rescaling with  $\Lambda$  and translating into the differential equation, we obtain

$$\frac{d}{dx} \left( (x + f_D/g)^2 \frac{d\Sigma(x)}{dx} \right) + \frac{N_{\text{cr}}^\infty}{4N} \Sigma(x) = 0, \quad (4.24)$$

$$\Sigma'(\sigma_m) = 0, \quad (\text{IR B.C.}) \quad (4.25a)$$

$$[(x + f_D/g)\Sigma]_{x=1}' = 0, \quad (\text{UV B.C.}) \quad (4.25b)$$

with the normalization condition being the same as Eq.(4.8c). The solution of Eq.(4.24) is easily obtained by substituting  $x$  into  $x + f_D/g$  in the solution of  $g \rightarrow \infty$  case, and the explicit form of the solution is given by

$$\Sigma(x) = \frac{\sigma_m^{\frac{1}{D-2}}}{\sin\left(\frac{\omega}{2}\delta\right)} \left( \frac{\sigma_m + f_D/g}{x + f_D/g} \right)^{\frac{1}{2}} \sin \left\{ \frac{\omega}{2} \left[ \ln \frac{x + f_D/g}{\sigma_m + f_D/g} + \delta \right] \right\}. \quad (4.26)$$

In Eq.(4.26)  $\omega$  and  $\delta$  are the same as those defined in Eq.(4.10), and we wrote down only the solution corresponding to the case of  $N < N_{\text{cr}}^\infty$ , because only this solution has a possibility to satisfy the UV boundary condition Eq.(4.25b). The UV boundary condition tells us the scaling relation

$$\frac{m}{\Lambda} = \left\{ (1 + f_D/g) e^{2\delta - \frac{2\pi}{\omega}} - f_D/g \right\}^{\frac{1}{D-2}} = (1 + g_{\text{cr}}/f_D)^{\frac{-1}{D-2}} \left[ 1 - \frac{g_{\text{cr}}}{g} \right]^{\frac{1}{D-2}}, \quad (4.27)$$

where,

$$g_{\text{cr}} \equiv f_D \left( e^{\frac{2\pi}{\omega} - 2\delta} - 1 \right) \quad (4.28)$$

is the critical coupling above which the dynamical mass is generated. From Eq.(4.27) the beta function of  $g$  can be seen as follows:

$$\beta_\Lambda(g) = \Lambda \frac{\partial g}{\partial \Lambda} = (D-2)g \left( 1 - \frac{g}{g_{\text{cr}}} \right), \quad (4.29)$$

which coincides with the one of the  $D$ -dimensional GN model<sup>8), 7)</sup> except for the expression of  $g_{\text{cr}}$ . At this moment it is not clear why these are the same to each other in contrast to the case of  $g \ll 1$  to be discussed in the next subsection. Furthermore the anomalous dimension is given as

$$\gamma_{\bar{\psi}\psi}(g) = (D-2) \frac{1 + f_D/g_{\text{cr}}}{1 + f_D/g}, \quad (4.30)$$

which is calculated by using

$$\langle \bar{\psi}\psi \rangle_\Lambda = - \frac{32N^2}{(4\pi)^{\frac{D}{2}} (D-2) \Gamma(\frac{D}{2}) N_{\text{cr}}^\infty} \cdot (\sigma_m + f_D/g) m \Lambda^{D-2} \quad (4.31)$$

obtained by substituting Eq.(4.26) into Eq.(4.18). Again  $\gamma_{\bar{\psi}\psi}(g \simeq g_{\text{cr}}) \simeq D-2$  in coincidence with GN model in  $D$  dimensions.<sup>8), 7)</sup>

#### 4.3. Analytical solution : $g \ll 1$ case

We finally analyze the case that the four fermion coupling constant is very small. In this case the integral defining the kernel function of the SD equation can easily be performed explicitly as

$$d(k^2)[D - \eta(k^2)] = G + O(G^2), \quad (4.32)$$

$$K(p, q; G) = \frac{2DG}{(4\pi)^{D/2} (D-2) \Gamma(D/2)} \equiv M_D G, \quad (4.33)$$

which reduces a momentum independent solution. Then the scaling relation is obtained as

$$\frac{m}{\Lambda} = \left(1 - \frac{g_{\text{cr}}}{g}\right)^{1/(D-2)}, \quad (4.34)$$

where

$$g_{\text{cr}} = \frac{N}{M_D} = \frac{(4\pi)^{D/2}(D-2)\Gamma(D/2)N}{2D} \quad (4.35)$$

is the critical coupling in this case, and the beta function and the anomalous dimension are obtained as

$$\beta_\Lambda(g) = (D-2)g \left(1 - \frac{g}{g_{\text{cr}}}\right), \quad (4.36)$$

$$\gamma_{\bar{\psi}\psi}(g) = (D-2)\frac{g}{g_{\text{cr}}}, \quad (4.37)$$

respectively, similarly to that of the  $D$ -dimensional GN model.<sup>8),7)</sup>

As a summary of this section let us notice that both  $g \gg 1$  and  $g \ll 1$  cases give the same result  $\gamma_{\bar{\psi}\psi} \simeq (D-2)$  at  $g \simeq g_{\text{cr}}$ , which is expected from the fact that the anomalous dimension near the critical line is a kind of the critical exponents. On the other hand, the result obtained in the infinite limit of the coupling constant is different from the others, since in this case the renormalization group functions are calculated along with the  $N$ -axis in  $(N, 1/g)$  plane. The similar situation can be seen in the gauged NJL model.<sup>7)</sup>

## §5. Numerical Study for the SD equation in (2+1) Dimensions

As indicated in the last section, we check the validity of the ansatz used to solve the SD equation for  $g \gg 1$ .

In this paper we would like to show the result only in (2+1) dimensions, in which the integral kernel can be calculated for any value of  $g$  as

$$\widetilde{K}(x, y; g) = \frac{8}{\pi^2 xy} [\kappa(x+y; g) - \kappa(|x-y|; g)], \quad (5.1)$$

where,

$$\kappa(z; g) \equiv \frac{1}{g} \left[ \frac{4gz}{3} + \frac{1}{gz} - \left(1 + \frac{1}{g^2 z^2}\right) \ln(1 + gz) \right]. \quad (5.2)$$

In the above equations, we already used variables normalized by the cut-off, which are used frequently in previous sections. Since the kernel is solved analytically, we can skip a few steps in the numerical calculation, which include complicated numerical integrations. Moreover, let us restrict to the linearized SD equation Eq.(3.10) instead of the original nonlinear integral

equation, since the linear integral equation can be treated as an eigenvalue problem which is easier to treat on computational analysis. As mentioned in §4, it is enough for investigating the structure of the critical line.

The figures following below show the result of the numerical calculation with the analytical solutions given in the previous section.

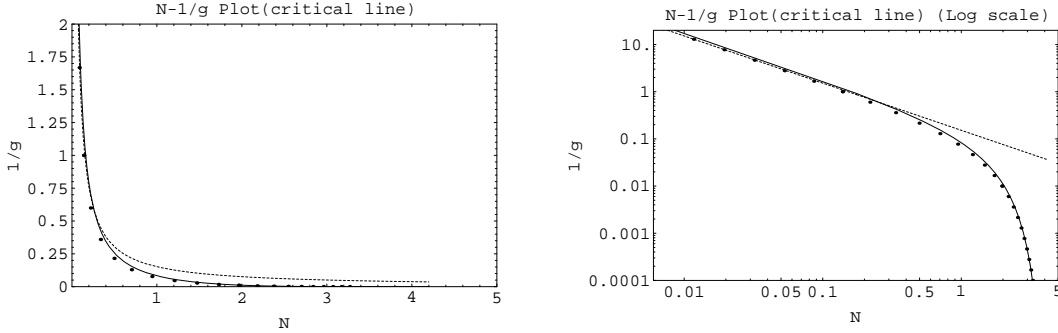


Fig. 1. The critical line in (2+1) dimensions

The plot points in Fig.1 are the result of numerical calculation with the kernel of Eq.(5.1). Lines represent the analytical solutions, a solid one is the solution of  $g \gg 1$  given in §4.2., and a dashed one is that of  $g \ll 1$  given in §4.3..

From these figures it is concluded that the analytical calculation with the ansatz well reproduces the critical line obtained by the numerical analysis.

## §6. Conclusion and Discussion

As an extension of our previous work,<sup>12)</sup> we have further studied dynamical fermion mass generation of the Thirring model as a gauge theory in  $D(2 < D < 4)$  dimensions.

After the introduction of the hidden local symmetry and the Schwinger-Dyson equation, we have completed the existence proof of a nontrivial solution of the SD equation made in Ref. 12) where an assumption on the property of eigenvalue of the integral equation was implicitly made. Furthermore we have shown that the critical line in  $(N, 1/g)$  plane starts from  $(0, \infty)$  and ends at  $(N_{\text{cr}}(\infty), 0)$ , which can be proven only by using the asymptotic behavior of the kernel with respect to  $g$ , the dimensionless four-fermion coupling constant.

As natural but nontrivial extensions of the previous work, we have found the analytical solutions of the linearized SD equation (bifurcation theory) in  $D(2 < D < 4)$  dimensions for some region of  $g$ .



In the case of  $g \rightarrow \infty$ , the SD equation in  $D$  dimensions can be solved exactly and the nontrivial solution have been obtained in the region  $N < N_{\text{cr}}^\infty (< \infty)$ , which is consistent with the result in (2+1) dimensions.<sup>12)</sup> This general  $D$ -dimensional form of the solution tells us that the critical value  $N_{\text{cr}}^\infty$  which depends only on  $D$  is monotonically decreasing for  $2 < D < 4$  from infinity ( $D \rightarrow 2$ ) to zero ( $D \rightarrow 4$ ), and reproduces the result in  $D = 3$  of Ref. 12). This result would imply that in  $D = 4$  dynamical mass generation does not occur even if we takes  $g \rightarrow \infty$ , while there is only the broken phase in  $D = 2$ . However, there are subtleties in both dimensions concerning the UV/IR divergences which we did not take into account because of the absence of those in  $2 < D < 4$  dimensions. As indicated in Appendix A, we should not believe the result of the linearized SD equation in  $D = 2$ , although the SD equation itself is exact for  $D = 2$ , since the IR singularity may break the validity of this equation. In case of  $D = 4$  the UV divergence should be properly regularized in the evaluation of the propagator of the dynamical gauge boson. Therefore we should study more carefully in this case.

The analytical solutions have been obtained also for finite  $g$  in the cases of  $g \gg 1$  and  $g \ll 1$ . These have given explicit form of the critical line which indicates a finite critical coupling constant  $g_{\text{cr}}$  for  $N < N_{\text{cr}}^\infty$  as opposed to the result of Ref. 11) claims  $g_{\text{cr}}$  for all  $N$ . On the other hand, at least qualitatively, our result seems to be consistent with that of Ref. 13) which is obtained also based on the HLS but through the inversion method instead of the SD equation. Furthermore, it should be emphasized that the numerical studies<sup>14), 15)</sup> would indicate that there is a finite critical value of  $N$ .

Moreover, we have found that the solutions obtained at both  $g \gg 1$  and  $g \ll 1$  gives the same expression for  $\beta_A(g)$  and the same value of the anomalous dimension on the critical line, i.e.,  $\gamma_m(g) = (D - 2)$  at  $g = g_{\text{cr}}$  which is identical to that of  $D$ -dimensional Gross-Neveu model.<sup>8)</sup>

Finally, we have confirmed validity of the ansatz used in the analytical calculation through numerical study of the linearized SD equation with the exact kernel, although limited to (2+1) dimensions. This result has shown that the numerical solution agrees with the analytical one in both qualitative and quantitative sense. However, it may be nontrivial whether or not we can extend this agreement to other dimensions, which should be confirmed by the efforts to the more complicated numerical studies.

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## Appendix A

### — Brief Summary of The Bifurcation Theory —

In this Appendix we will briefly summarize the bifurcation theory, and comment on a possibility of break down of the bifurcation theory in (1+1) dimensions.

The bifurcation theory is applied to such a following functional equation,

$$\phi(x) = \lambda F[\phi; x], \quad (\text{A}\cdot 1)$$

where,  $F[\phi; x]$  in the R.H.S. is a functional of  $\phi(x)$  as well as a function of  $x$ , and  $\lambda$  is a parameter. Here let us assume that one of the solutions of Eq.(A.1) is already known, which is denoted as  $\phi_0(x; \lambda)$ . Hence we are interested in the solution which bifurcates from  $\phi_0(x; \lambda)$  continuously at  $\lambda = \lambda_{\text{br}}$ .

As discussed in Ref. 29), the necessary condition for the existence of such a bifurcated solution is as follows. *The functional  $F[\phi; x]$  is functionally differentiable in the third order with respect to  $\phi(x)$  at  $\phi(x) = \phi_0(x; \lambda_{\text{br}})$ , where the bifurcation point  $\lambda_{\text{br}}$  is determined as the eigenvalue of the linearized equation for  $\delta\phi(x)$ :*

$$\delta\phi(x) = \lambda \int dy \left. \frac{\delta F[\phi; x]}{\delta\phi(y)} \right|_{\phi(x)=\phi_0(x)} \delta\phi(y). \quad (\text{A}\cdot 2)$$

Thus the bifurcation solution is given as  $\phi(x) = \phi_0(x; \lambda_{\text{br}}) + \delta\phi(x)$  in the vicinity of  $\lambda_{\text{br}}$ .

We can easily check with the SD equation (3.7b) that the R.H.S. satisfies the necessary condition. Moreover, we can also confirm that the necessary condition is satisfied even if we use the fermion full propagator when calculating the dynamical gauge boson's propagator and the integral kernel of Eq.(3.7b), although the kernel cannot be solved explicitly in such a case.

Concerning with the above statement, we should comment on the case of  $D = 2$  in which the exact solution was found to be equivalent to a free theory having no dimensionful parameter. However, if we solve Eq.(3.10) in  $D = 2$  naively, a nontrivial solution for  $g \geq 0$  exists, which is constant for the momentum as  $\delta B(p^2) \equiv m = \Lambda \exp\{-N(1 + \pi/g)\}$ . It should be emphasized that the ladder approximation is not wrong in this case, although it has been usually suspected. The reason is that the set of vertex functions

$$\begin{aligned} S(p) &= i/\not{p}, \quad D_{\mu\nu}(k) = (g^{-1} + \pi^{-1})g_{\mu\nu}, \quad \Gamma^\mu(p, q) = \gamma^\mu, \\ \text{other vertex function} &= 0 \end{aligned} \quad (\text{A}\cdot 3)$$

is the exact solution of the full series of the SD equations, which means that the ladder approximation is *exact* in (1+1) dimensions, if the bifurcation theory is valid.

Do these considerations conclude the dynamical mass generation in the (1+1)-dimensional Thirring model? This conffiction will be resolved by investigating the integral kernel more in detail. Thus it might be shown that we are not able to apply the bifurcation theory to the case of  $D = 2$  because of the non-analyticity of  $K(p, q; G)$  with respect to the mass function  $B(p^2)$  near a trivial solution  $B(p^2) \equiv 0$ , i.e.,  $\delta K(p, q; G)/\delta B(p^2)|_{B(p^2) \equiv 0} = \infty$ . This property of the kernel is contrary to the necessary condition explained above. Therefore in (1+1) dimension we should not naively conclude the existence of a nontrivial solution through the bifurcation theory.

## Appendix B

— *Proof that  $\lambda(g, \sigma_m)$  is a decreasing function of  $\sigma_m$*  —

In §3 we needed to prove that the maximal eigenvalue of the integral equation (3.14) is monotonically decreasing function with respect to  $\sigma_m$ . For this purpose, we will prove more general statement in this Appendix.

Let us consider the following eigenvalue problem:

$$\phi(x) = \frac{1}{\lambda} \int_a^b dy K(x, y) \phi(y), \quad (\text{B}\cdot 1)$$

where  $K(x, y)$  is a real symmetric integral kernel which is finite on the region  $a \leq x, y \leq b$ . We denote eigenvalues and corresponding eigenfunctions as  $\lambda_n(a)$  and  $\phi_n(x; a)$ , in which  $n$  is assigned as  $\lambda_1(a) > \lambda_2(a) > \dots$ . The dependence of  $a$  is defined by both Eq.(B.1) and the normalization condition

$$\int_a^b dx \phi_m(x; a) \phi_n(x; a) = \delta_{mn}, \quad (\text{B}\cdot 2)$$

and hence  $\lambda_n(a)$  is expressed as

$$\lambda_n(a) = \int_a^b \int_a^b dx dy \phi_n(x; a) K(x, y) \phi_n(y; a). \quad (\text{B}\cdot 3)$$

By using this explicit expression of  $\lambda_n(a)$ , we can calculate the derivative of  $\lambda_n(a)$  with respect to  $a$  as

$$\begin{aligned} \frac{d\lambda_n(a)}{da} &= -2\phi_n(a; a) \int_a^b dy K(a, y) \phi_n(y; a) \\ &\quad + 2 \int_a^b \int_a^b dx dy \frac{\partial \phi_n(x; a)}{\partial a} K(x, y) \phi_n(y; a) \\ &= -2\lambda_n(a) (\phi_n(a; a))^2 + 2\lambda_n(a) \int_a^b dx \frac{\partial \phi_n(x; a)}{\partial a} \phi_n(x; a), \end{aligned} \quad (\text{B}\cdot 4)$$

where, we use the symmetry property of  $K(x, y)$ . The second term of the last line of Eq.(B·4) is derived by the differentiation of Eq.(B·2):

$$\begin{aligned} 0 &= \frac{d}{da} \int_a^b dx \phi_n(x; a) \phi_n(x; a) \\ &= -(\phi_n(a; a))^2 + 2 \int_a^b dx \frac{\partial \phi_n(x; a)}{\partial a} \phi_n(x; a). \end{aligned} \quad (\text{B} \cdot 5)$$

Substituting Eq.(B·5) into Eq.(B·4), we finally obtain the formula as

$$\frac{d\lambda_n(a)}{da} = -\lambda_n(a) (\phi_n(a; a))^2 \begin{cases} \leq 0 & (\lambda_n(a) \geq 0) \\ > 0 & (\lambda_n(a) < 0) \end{cases}, \quad (\text{B} \cdot 6)$$

which is the statement that should be proven.

Combining with the fact that the maximal eigenvalue  $\lambda_1(a)$  has always positive sign, we have actually proven that  $\lambda_1(a)$  is monotonically decreasing function for the lower bound of the integral.

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